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Asymptotic wave propagation in a non-Newtonian compressible fluid with small dissipation

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Abstract. The spherical motion of a non-Newtonian compressible fluid is considered and a reductive perturbation method is used to study the point-explosion problem. The material response functions involved in the model under consideration are assumed to be of polynomial form and the resulting Burgers-like equation which governs the far-field approximation is investigated. A qualitative analysis of this equation is made via a numerical integration.

Keywords: non-Newtonian compressible fluid, asymptotic wave propagation, similarity solutions, point explosion

1. Introduction

As is well known, many fluids used in industrial applications do not behave according to the Newtonian constitutive relations. Such fluids, called non-Newtonian, are characterized by a nonlinear relation between stress and strain-rate tensors.

As far as we know, much effort has been devoted to study non-Newtonian incompressible fluids through a modified Newtonian law of viscosity in order to allow the viscosity to vary with the strain rate [1–2], namely

$$\sigma_{ij} = 2\eta\gamma_{ij}, \qquad \gamma_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \tag{1.1}$$

σ_{ij} and γ_{ij} begin the components of the viscous stress and the strain-rate tensors, respectively; unlike the classical Newtonian fluid, the viscosity coefficient $\eta = \eta(\mathrm{II}, \mathrm{III})$ depends on the invariants of the strain-rate tensor

$$\mathrm{I} = \gamma_{ii}, \qquad \mathrm{II} = \gamma_{ij}\gamma_{ji}, \qquad \mathrm{III} = \gamma_{ij}\gamma_{jk}\gamma_{ki}, \tag{1.2}$$

where the dependence on I does not exist for the incompressible case.

In the present paper we consider a compressible non-Newtonian fluid characterized by the following constitutive law

$$\sigma_{ij} = \alpha\gamma_{ij} + \beta\delta_{ij}, \tag{1.3}$$

where α and β depend on the invariants (1.2) and on the thermodynamic variables. Such an assumption appears to be a natural generalization of the Cauchy–Poisson Law of the classical Navier–Stokes theory of viscous fluids and it characterizes a particular class of the so-called Stokesian fluid [3, pp. 160–162].

Furthermore, when $\beta = 0$, the constitutive law (1.3) reduces to (1.1), *i.e.* to the case of an incompressible non-Newtonian fluid. Of course α must β and obey the restrictions imposed by the Clausius–Duhem inequality.

The model under consideration, namely that of compressible non-Newtonian fluids (in particular gases without bulk viscosity), could apply to many physical problems. For example, it may serve to model from a macroscopic point of view a gas in which long-chain molecules are dissolved provided that we limit ourselves to considering the problem in the framework of the hydrodynamics of a real fluid in which dissipative processes are taken into account. Of course, the gas gives the medium compressibility, while chain interactions provide the complexity of non-Newtonian behaviour. The model outlined above encompasses the macroscopic description of a number of real applications such as farmaceutical aerosol, environmental aerosol (sandstorm, soot) and industrial aerosol (smog, fuel). Furthermore we remark that the model under investigation belongs to the general class of multipolar viscous fluids (monopolar viscous fluid) (see [4] and the references quoted therein).

Our main goal is to investigate the point-explosion problem in a non-Newtonian fluid. As is well known [5–6], a violent explosion occurs when a large amount of explosion energy is concentrated in a small portion of a material medium which consequently begins to expand rapidly. The rapid explosion produces a disturbance headed by a strong shock wave called a blast wave propagating into the surrounding medium. The point explosion model may simulate many real problems as the mechanism of the supernova event (see [6] and the reference therein quoted).

Since we are essentially concerned with the study of a dissipative system, we assume the dissipation coefficients α and β to be small and of the same order ε with $\varepsilon \ll 1$. More precisely, we consider the solution of the system under consideration with vanishingly small dissipative coefficients and we assume that a shock is present in the limiting case when the fluid is perfect. In fact, a small dissipation is able to prevent the nonlinear breaking of the wave profile and, taking into account such a phenomenon, we are led to consider more general systems of equations where higher-order space derivatives are involved multiplied by small coefficients.

Unfortunately, in the case under consideration it is not possible to find in general an exact analytical solution, since the flow is not self-similar. However, in the limiting case $\varepsilon = 0$ (i.e. vanishing dissipative effects), if a strong explosion takes place in a point, the resulting motion is self-similar and it can be described by a similarity solution known as 'Sedov solution' which is in fact invariant with respect to an infinitesimal stretching group of transformations [7–9]. Hence it is natural to search for the solution of the full governing system under the form of an asymptotic development around the 'Sedov solution'.

Within the theoretical framework outlined above, in Section 2 we require the hyperbolic model associated with the full governing system to be invariant with respect to a stretching group of transformations; later, as it is usual for dissipative systems, we will search for a solution to the governing system in the form of an asymptotic expansion about a similarity solution of the hyperbolic model under consideration and we deduce a nonlinear evolution equation governing the far-field approximation. This equation is different from the usual Burgers-like equation, because of the occurrence therein of a series-like coefficient of the second-order derivative.

Next, in Section 3 we calculate the coefficients of this evolution equation in the case of the Sedov solution. In Section 4 we assume that the constitutive coefficients α and β involved in (1.3) are of polynomial form, so that the series-like coefficient of the corresponding evolution equation becomes a polynomial with respect to the first-order derivatives. For the resulting modified Burgers-like equation, in Section 5 we carry on a similarity analysis and we obtain some classes of solutions which are invariant with respect to a suitable infinitesimal group of transformations.

Finally, for a particular invariance transformation we try a numerical integration for the ordinary equation occurring in this case to investigate the wave profiles which are compared with the classical ones.

2. Basic equations and asymptotic analysis

We consider the spherical symmetric motion of a non-Newtonian fluid governed by the equations

$$\begin{aligned} &\rho_t + \rho u_r + u\rho_r = -\frac{2}{r}u\rho \\ &\rho(u_t + uu_r) = -p_r + \alpha\left(u_{rr} + \frac{2}{r}u_r - \frac{2}{r^2}u\right) + u_r\alpha_r + \beta_r \\ &\rho(e_t + ue_r) = -p\left(u_r + \frac{2}{r}u\right) + \alpha\left(u_r^2 + \frac{2}{r^2}u^2\right) + \beta\left(u_r + \frac{2}{r}u\right), \end{aligned} \tag{2.1}$$

where ρ is the mass density, u the flow speed, e the specific internal energy, p is the pressure, while r and t represent the radial and time coordinates, respectively. A subscript means differentiation with respect to the indicated variable.

Furthermore, the invariants of the strain-rate tensor become

$$\mathrm{I} = u_r + \frac{2}{r}u, \qquad \mathrm{II} = u_r^2 + \frac{2}{r^2}u^2, \qquad \mathrm{III} = u_r^3 + \frac{2}{r^3}u^3. \tag{2.2}$$

Since dissipative effects occur, we assume the dissipative coefficients α and β to be small and of the same order ε, that is

$$\alpha = \varepsilon\overline{\alpha}, \qquad \beta = \varepsilon\overline{\beta}, \quad \varepsilon \ll 1. \tag{2.3}$$

Taking into account (2.3), we observe that the system (2.1) may be rewritten in the following matrix form

$$\boldsymbol{U}_t + A(\boldsymbol{U})\boldsymbol{U}_r = \boldsymbol{B}(\boldsymbol{U}, r) + \varepsilon\{\boldsymbol{M}(\boldsymbol{U}, \boldsymbol{U}_r, r) + N(\boldsymbol{U}, \boldsymbol{U}_r, r)\boldsymbol{U}_{rr}\}, \tag{2.4}$$

where

$$\begin{aligned} &\boldsymbol{U} = \begin{bmatrix} \rho \\ u \\ e \end{bmatrix}, \qquad A = \begin{bmatrix} u, \rho, 0 \\ \frac{1}{\rho}p_\rho, u, \frac{1}{\rho}p_e \\ 0, \frac{p}{\rho}, u \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} -\frac{2}{r}u\rho \\ 0 \\ -\frac{2}{r}u\frac{p}{\rho} \end{bmatrix}, \\ &\boldsymbol{M} = \begin{bmatrix} 0 \\ m_{21} \\ \frac{\overline{\beta}}{\rho}\left(u_r + \frac{2}{r}u\right) + \frac{\overline{\alpha}}{\rho}\left(u_r^2 + \frac{2}{r^2}u^2\right) \end{bmatrix}, \qquad N = \begin{bmatrix} 0, 0, 0 \\ 0, \frac{1}{\rho}\frac{\partial}{\partial u_r}(\overline{\beta} + u_r\overline{\alpha}), 0 \\ 0, 0, 0 \end{bmatrix} \end{aligned} \tag{2.5}$$

with

$$m_{21} = \left(\frac{\partial\overline{\beta}}{\partial\rho} + u_r\frac{\partial\overline{\alpha}}{\partial\rho}\right)\frac{\rho_r}{\rho} + \left(\frac{\partial\overline{\beta}}{\partial e} + u_r\frac{\partial\overline{\alpha}}{\partial e}\right)\frac{e_r}{e} + \frac{2}{\rho r}\left(u_r - \frac{u}{r}\right)$$
$$\times\left(\overline{\alpha} + \frac{\partial\overline{\beta}}{\partial \mathrm{I}} + u_r\frac{\partial\overline{\alpha}}{\partial \mathrm{I}} + \frac{2u}{r}\left(\frac{\partial\overline{\beta}}{\partial \mathrm{II}} + u_r\frac{\partial\overline{\alpha}}{\partial \mathrm{II}}\right) + \frac{3}{r^2}u^2\left(\frac{\partial\overline{\beta}}{\partial \mathrm{III}} + u_r\frac{\partial\overline{\alpha}}{\partial \mathrm{III}}\right)\right).$$

In order that we may study the influence of the dissipative effects on the point-explosion problem by using an asymptotic expansion around a similarity solution like the Sedov solution, we require first of all the hyperbolic system associated with (2.4) to be invariant with respect to the following group of stretching transformations [7–9]

$$r^* = \lambda r, \qquad t^* = \lambda^{\gamma} t, \qquad \rho^* = \lambda^{a}\rho, \qquad u^* = \lambda^{1-\gamma}u, \qquad e^* = \lambda^{2(1-\gamma)}e, \tag{2.6}$$

where a and γ are the similarity exponents. The invariance condition with respect to the group (2.6) implies for the pressure p the following constitutive law

$$p = \rho^{2(1-\gamma)/a+1}P(\zeta) \quad \text{with} \quad \zeta = e\rho^{2(\gamma-1)/a}. \tag{2.7}$$

By considering the 'canonical variables' [10, pp. 31–33]

$$\begin{aligned} &\tau = \log t, \qquad \xi = \frac{r}{t^{1/\gamma}}, \\ &\rho = t^{a/\gamma}R(\xi,\tau), \qquad u = t^{(1-\gamma)/\gamma}V(\xi,\tau), \qquad e = t^{2(1-\gamma)/\gamma}E(\xi,\tau), \end{aligned} \tag{2.8}$$

we see that the hyperbolic system associated with (2.4) becomes

$$\boldsymbol{W}_{\tau} + \left(\widehat{A}(\boldsymbol{W}) - \frac{\xi}{\gamma}I\right)\boldsymbol{W}_{\xi} = \widehat{\boldsymbol{B}}(\boldsymbol{W},\xi), \tag{2.9}$$

where I is the identity matrix and

$$\boldsymbol{W} = \begin{bmatrix} R \\ V \\ E \end{bmatrix}, \quad \widehat{A} = \begin{bmatrix} V,\ R,\ 0 \\ \widehat{a}_{21},\ V,\ P' \\ 0,\ R^{2(1-\gamma)/a}P,\ V \end{bmatrix}, \qquad \widehat{\boldsymbol{B}} = \begin{bmatrix} -\left(\dfrac{a}{\gamma} + 2\dfrac{V}{\xi}\right)R \\ \dfrac{\gamma-1}{\gamma}V \\ \dfrac{2(\gamma-1)}{\gamma}E - 2PR^{2(1-\gamma)/a}\dfrac{V}{\xi} \end{bmatrix}, \tag{2.10}$$

with

$$\widehat{a}_{21} = \frac{a+2(1-\gamma)}{a}PR^{(2(1-\gamma)-a)/a} + \frac{2(\gamma-1)}{a}\frac{E}{R}P',$$

where here and in what follows the ‘ $'$ ’ denotes the derivative of the function with respect to its argument.

In what follows we look for a solution of the system (2.4) of the form [11, 12]

$$\boldsymbol{U} = \boldsymbol{U}_0(r, t) + \sum_{k=1}^{\infty} \varepsilon^{k/2} \boldsymbol{U}_k(r, t, \phi), \tag{2.11}$$

where $\phi = \varepsilon^{-1/2}\varphi(\xi, \tau)$, $\boldsymbol{U}_0 = T^{(0)}\boldsymbol{W}_0$ is a particular solution of the hyperbolic system associated with (2.4) and $\boldsymbol{U}_k = T^{(k)}\boldsymbol{W}_k$, being

$$T^{(0)} = \begin{bmatrix} t^{a/\gamma}, 0, 0 \\ 0, t^{(1-\gamma)/\gamma}, 0 \\ 0, 0, t^{2(1-\gamma)/\gamma} \end{bmatrix}, \qquad T^{(k)} = \begin{bmatrix} t^{\delta_k}, 0, 0 \\ 0, t^{\mu_k}, 0 \\ 0, 0, t^{\nu_k} \end{bmatrix} \tag{2.12}$$

and $\boldsymbol{W}_0 = \boldsymbol{W}_0(\xi, \tau)$ is a solution of (2.9), while $\boldsymbol{W}_k = \boldsymbol{W}_k(\xi, \tau, \phi)$ and δ_k, μ_k, ν_k, are arbitrary constants to be determined by our analysis in order that the different terms in the expansion (2.11) exhibit a similarity form.

Taking into account (1.9), we have to consider the following expansions

$$\begin{aligned} &A = A_0 + \varepsilon^{1/2}(\nabla_{\boldsymbol{U}} A)_0 \boldsymbol{U}_1 + O(\varepsilon), \qquad \boldsymbol{B} = \boldsymbol{B}_0 + \varepsilon^{1/2}(\nabla_{\boldsymbol{U}} \boldsymbol{B})_0 \boldsymbol{U}_1 + O(\varepsilon), \\ &\boldsymbol{M} = \boldsymbol{M}_0 + \sum_{k=1}^{\infty} \frac{\varphi_r^k}{k!}([\boldsymbol{U}_{1\phi}^T \cdot \nabla_{\boldsymbol{U}_r}]^{(k)} \boldsymbol{M})_0 + O(\varepsilon^{1/2}), \\ &N = N_0 + \sum_{k=1}^{\infty} \frac{\varphi_r^k}{k!}([\boldsymbol{U}_{1\phi}^T \cdot \nabla_{\boldsymbol{U}_r}]^{(k)} N)_0 + O(\varepsilon^{1/2}), \end{aligned} \tag{2.13}$$

where the subscript ‘$_0$’ means that the quantity is evaluated at $\boldsymbol{U} = \boldsymbol{U}_0$.

By inserting (2.11), (2.13) into (2.4) and cancelling the coefficients of ε^0 and $\varepsilon^{1/2}$, we obtain, respectively

$$\varphi_\xi T^{(0)} \left(\widehat{A}_0 + \left(\frac{\varphi_\tau}{\varphi_\xi} - \frac{\xi}{\gamma} \right) I \right) T \frac{\partial \boldsymbol{W}_1}{\partial \phi} = 0, \tag{2.14}$$

$$\begin{aligned} &\varphi_\xi \left(\widehat{A}_0 + \left(\frac{\varphi_\tau}{\varphi_\xi} - \frac{\xi}{\gamma} \right) I \right) \widehat{T} \frac{\partial \boldsymbol{W}_2}{\partial \phi} + \frac{\partial (T\boldsymbol{W}_1)}{\partial \tau} + \left(\widehat{A}_0 - \frac{\xi}{\gamma} I \right) T \frac{\partial \boldsymbol{W}_1}{\partial \xi} \\ &\quad + \varphi_\xi (\nabla_{\boldsymbol{W}} \widehat{A} T \boldsymbol{W}_1)_0 T \frac{\partial \boldsymbol{W}_1}{\partial \phi} + (\nabla_{\boldsymbol{W}} \widehat{A} T \boldsymbol{W}_1)_0 \frac{\partial \boldsymbol{W}_0}{\partial \xi} - (\nabla_{\boldsymbol{W}} \widehat{\boldsymbol{B}} T \boldsymbol{W}_1)_0 \\ &= t^{(\gamma-2)/\gamma} \varphi_\xi^2 \sum_{k=0}^{\infty} \frac{\varphi_\xi^k}{k!} t^{-k/\gamma} (T^{(0)})^{-1} \\ &\quad \times \left(\left[\left(T^{(1)} \frac{\partial \boldsymbol{W}_1}{\partial \phi} \right)^T \cdot \nabla_{\boldsymbol{U}_r} \right]^{(k)} N \right)_0 T^{(1)} \frac{\partial^2 \boldsymbol{W}_1}{\partial \phi^2}, \end{aligned} \tag{2.15}$$

where $T = (T^{(0)})^{-1}T^{(1)}$, $\widehat{T} = (T^{(0)})^{-1}T^{(2)}$ and the following relations hold

$$A_0 = t^{(1-\gamma)/\gamma} T^{(0)} \widehat{A}_0 (T^{(0)})^{-1}, \qquad \boldsymbol{B}_0 = t^{-1}\left(T^{(0)}\widehat{\boldsymbol{B}}_0 + \frac{\partial T^{(0)}}{\partial \tau}\boldsymbol{W}_0\right),$$
$$(\nabla_{\boldsymbol{U}} A)_0 T^{(1)} \boldsymbol{W}_1 = t^{(1-\gamma)/\gamma} T^{(0)} (\nabla_{\boldsymbol{W}} \widehat{A} T \boldsymbol{W}_1)_0 (T^{(0)})^{-1}, \tag{2.16}$$
$$(\nabla_{\boldsymbol{U}} \boldsymbol{B})_0 T^{(1)} \boldsymbol{W}_1 = t^{-1}\left(T^{(0)}(\nabla_{\boldsymbol{W}} \widehat{\boldsymbol{B}} T \boldsymbol{W}_1)_0 + \frac{\partial T^{(0)}}{\partial \tau} T \boldsymbol{W}_1\right).$$

A direct inspection of (2.14) shows that $\partial \boldsymbol{W}_1/\partial \phi \neq 0$, provided the phase function $\varphi(\xi, \tau)$ is a solution of the characteristic equation

$$\det\left(\widehat{A}_0 - \frac{\xi}{\gamma} I + \frac{\varphi_\tau}{\varphi_\xi} I\right) = 0, \tag{2.17}$$

which yields

$$\lambda_1 = V_0 - \frac{\xi}{\gamma}, \qquad \lambda_{2,3} = V_0 - \frac{\xi}{\gamma} \pm D_0 \tag{2.18}$$

$\lambda = -(\varphi_\tau/\varphi_\xi)$ being an eigenvalue of the matrix $(\widehat{A}_0 - (\xi/\gamma)I)$, while

$$D_0^2 = P_0 R_0^{2(1-\gamma)/a}\left(P_0' + \frac{a + 2(1-\gamma)}{a}\right) + \frac{2(\gamma - 1)}{a} E_0 P_0', \tag{2.19}$$

provided

$$\mu_1 - \frac{1-\gamma}{\gamma} = \delta_1 - \frac{a}{\gamma} = \nu_1 - \frac{2(1-\gamma)}{\gamma}.$$

The system (2.14) yields

$$\boldsymbol{W}_1 = \pi(\xi, \tau, \phi)\boldsymbol{d}_{0'} \tag{2.20}$$

$\boldsymbol{d}_0$ being the right eigenvector of the matrix $(\widehat{A}_0 - (\xi/\gamma)I)$ corresponding to the eigenvalue λ.

Now inserting (2.20) into (2.15) and multiplying the resulting equation by the left eigenvector $\boldsymbol{l}_0$, we have

$$\frac{\partial \pi}{\partial \sigma} + \left((\mu_1 - \mu_0)(\boldsymbol{l}\cdot\boldsymbol{d})_0 + \left(\boldsymbol{l}\cdot\frac{\partial \boldsymbol{d}}{\partial \sigma}\right)_0 + (\boldsymbol{l}(\nabla_{\boldsymbol{W}}\widehat{A}\cdot\boldsymbol{d}))_0 \frac{\partial \boldsymbol{W}_0}{\partial \xi}\right.$$
$$\left. - (\boldsymbol{l}(\nabla_{\boldsymbol{W}}\widehat{\boldsymbol{B}}\cdot\boldsymbol{d})_0)\right)\frac{\pi}{(\boldsymbol{l}\cdot\boldsymbol{d})_0} + \varphi_\xi t^{\mu_1-\mu_0}(\nabla\lambda\cdot\boldsymbol{d})_0 \pi \frac{\partial \pi}{\partial \phi}$$
$$= \frac{t^{1-2/\gamma}}{(\boldsymbol{l}\cdot\boldsymbol{d})_0}\varphi_\xi^2 \sum_{k=0}^{\infty}\frac{\varphi_\xi^k}{k!} t^{-k/\gamma}\left(\frac{\partial \pi}{\partial \phi}\right)^k (\boldsymbol{l}([(T^{(1)}\boldsymbol{d})^T\cdot\nabla_{\boldsymbol{U}_r}]^{(k)} N)\boldsymbol{d})_0 \frac{\partial^2 \pi}{\partial \phi^2}, \tag{2.21}$$

where $\partial/\partial\sigma = \partial/\partial\tau + \lambda(\partial/\partial\xi)$ represents the derivative along the characteristic rays

$$\frac{d\xi}{d\tau} = \lambda, \quad \tau = \sigma, \tag{2.22}$$

associated to the variable $\varphi(\xi,\tau)$ determined by $\varphi_\tau + \lambda(\boldsymbol{U}_0(\xi,\tau),\xi)\varphi_\xi = 0$.

Now we limit ourselves to considering the eigenvalue λ_2 to which correspond the right and left eigenvectors

$$(\boldsymbol{l}_2)_0 = \left[\frac{D_0^2 - P_o P_0' R_0^{2(1-\gamma)/a}}{R_0}, D_0, P_0'\right], \qquad (\boldsymbol{d}_2)_0 = \begin{bmatrix} R_0 \\ D_0 \\ P_0 R_0^{2(1-\gamma)/a} \end{bmatrix} \tag{2.23}$$

so that Equation (2.21) reduces to

$$\frac{\partial \pi}{\partial \sigma} + f(\sigma)\pi + g(\sigma)\pi\frac{\partial \pi}{\partial \phi} = \left(\sum_{k=0}^{\infty} a_k(\sigma)\left(\frac{\partial \pi}{\partial \phi}\right)^k\right)\frac{\partial^2 \pi}{\partial \phi^2} \tag{2.24}$$

with

$$\begin{aligned}
f(\sigma) \;=\; & \frac{1}{2D_0^2}\Bigg(\frac{D_0^2 - P_0 P_0' R_0^{2(1-\gamma)/a}}{R_0}\frac{\partial R_0}{\partial \sigma} + D_0\frac{\partial D_0}{\partial \sigma} + P_0'\frac{\partial}{\partial \sigma}(P_0 R_0^{2(1-\gamma)/a}) \\
& + \left(\delta_1 + 2\frac{V_0}{\xi} + 2\frac{D_0}{\xi}\right)(D_0^2 - P_0 P_0' R_0^{2(1-\gamma)/a}) + \mu_1 D_0^2 + \nu_1 P_0 P_0' R_0^{2(1-\gamma)/a} \\
& + P_0'\Bigg(\frac{4(1-\gamma)}{a}P_0\frac{V_0}{\xi}R_0^{2(1-\gamma)/a} + 2D_0\frac{P_0}{\xi}R_0^{2(1-\gamma)/a} + 2P_0P_0'\frac{V_0}{\xi}R_0^{2(1-\gamma)/a} \\
& + \frac{4(\gamma-1)}{a}E_0\frac{V_0}{\xi}P_0'\Bigg) + D_0\frac{\partial R_0}{\partial \xi}\Bigg(\frac{D_0^2 - P_0 R_0^{2(1-\gamma)/a}}{R_0} + \frac{2(\gamma-1)}{a}\frac{E_0}{R_0}P_0P_0'' \\
& + \frac{4(1-\gamma)^2}{a^2}\frac{E_0^2 P_0'' R_0^{2(\gamma-1)/a} + P_0 R_0^{2(1-\gamma)/a} - P_0' E_0}{R_0}\Bigg) \\
& + \frac{\partial V_0}{\partial \xi}\Bigg(2D_0^2 + P_0(P_0')^2 R_0^{2(1-\gamma)/a} - \frac{a + 2(\gamma-1)}{a}P_0P_0'R_0^{2(1-\gamma)/a} \\
& + \frac{2(\gamma-1)}{a}E_0(P_0')^2\Bigg) + D_0\frac{\partial E_0}{\partial \xi}\left(P_0P_0'' + P_0' + \frac{2(\gamma-1)}{a}E_0P_0''R_0^{2(\gamma-1)/a}\right)\Bigg),
\end{aligned} \tag{2.25}$$

$$\begin{aligned}
g(\sigma) \;=\; & \frac{\varphi_\xi\, \mathrm{e}^{(\mu_1-\mu_0)\sigma}}{2D_0}\Bigg(2D_0^2 + \frac{2(\gamma-1)}{a}E_0(P_0')^2 + \frac{4(\gamma-1)}{a}E_0P_0P_0'' \\
& + \frac{2(\gamma-1)(a+2(1-\gamma))}{a^2}E_0P_0' + \frac{4(\gamma-1)^2}{a^2}P_0''E_0^2R_0^{2(\gamma-1)/a} \\
& + \frac{a+2(1-\gamma)}{a}P_0R_0^{2(1-\gamma)/a}\left(\frac{2(1-\gamma)}{a} + P_0'\right) + P_0R_0^{2(1-\gamma)/a}((P_0')^2 + P_0'')\Bigg),
\end{aligned}$$

$$a_k(\sigma) = \frac{1}{2R_0}\,\mathrm{e}^{[(\gamma-2-a)/\gamma + k(\mu_1 - 1/\gamma)]\sigma}\frac{\varphi_\xi^{2+k}}{k!}D_0^k\left(\frac{\partial^{k+1}}{\partial u_r^{k+1}}(u_r\overline{\alpha} + \overline{\beta})\right)_0.$$

The occurrence of the series-like coefficient of $\partial^2\pi/\partial\phi^2$ in (2.24) is due to the dependence of the matrix N in (2.4) upon $\boldsymbol{U}_r$. An evolution equation of this form has already been obtained in a previous work concerning the plain vibrations of a moving threadline [13]. In the present paper we are interested in possible constitutive laws for $\overline{\alpha}$ and $\overline{\beta}$ which allow the above-mentioned series to converge.

From $(2.25)_3$ it is easily seen that it is possible to truncate the series occurring in (2.24) in the following cases

$$\text{(i)} \quad \frac{\partial^{k+1}}{\partial u_r^{k+1}}(u_r\overline{\alpha}+\overline{\beta}) = 0, \quad \forall k \geqslant m, \quad m \in \mathbb{N},$$

so that we obtain

$$u_r\overline{\alpha}+\overline{\beta} = P_m(u_r), \tag{2.26}$$

where $P_m(u_r)$ represents a polynomial of degree m with respect to u_r whose coefficients depend on the thermodynamic variables ρ and e. As is obvious in this case, the coefficient of $\partial^2\pi/\partial\phi^2$ becomes a polynomial of degree m with respect to $\partial\pi/\partial\phi$.

$$\text{(ii)} \quad \frac{\partial^{k+1}}{\partial u_r^{k+1}}(u_r\overline{\alpha}+\overline{\beta}) = GF^{k+1}, \quad \forall k \in \mathbb{N},$$

which implies the following consitutive relations

$$u_r\overline{\alpha}+\overline{\beta} = E\exp\{Fu_r\}+\widetilde{E}, \qquad G = E\exp\{Fu_r\} \tag{2.27}$$

with E, F and $\widetilde{E}$ arbitrary functions depending on ρ, e and u.

If condition (ii) holds, the series under consideration converges to an exponential function, that is

$$\frac{1}{2R_0}\,\mathrm{e}^{((\gamma-2-a)/\gamma)\sigma}E_0F_0\varphi_\xi^2\exp\left\{F_0\left(u_r+\varphi_\xi D_o\,\mathrm{e}^{(\mu_1-1/\gamma)\sigma}\frac{\partial\pi}{\partial\phi}\right)\right\}. \tag{2.28}$$

If $\overline{\alpha}$ and $\overline{\beta}$ are given functions of their arguments satisfying (2.26) or (2.27), then the series occurring in the evolution equation (2.24) converges. However, since (2.26) or (2.27) represents one condition to be satisfied by two response functions, if in a problem under considerations $\overline{\alpha}$ and $\overline{\beta}$ are not specified *a priori*, these conditions can be used in order to select classes of models of form (2.26) or (2.27), allowing the series occurring in (2.24) to terminate. Such an aspect can be of certain interest because, as far as we know, much effort has been devoted to the study of incompressible non-Newtonian fluids, so that, we have found in the literature very little information about the constitutive function $\overline{\beta}$ related to the compressibility of the fluid.

A further criterion to determine the functional forms of $\overline{\alpha}$ and $\overline{\beta}$ which are physically meaningful, is to require that they satisfy the Clausius–Duhem inequality which in our case reduces to

$$\sigma_{ij}\gamma_{ij} \geqslant 0 \tag{2.29}$$

for every possible thermodynamical process. This will be considered later on.

As usual, we consider the following transformation of variables [14]

$$q = \pi \exp\left\{\int f(\sigma)\,\mathrm{d}\sigma\right\}, \qquad \sigma^* = \int g(\sigma) \exp\left\{-\int f(\sigma)\,\mathrm{d}\sigma\right\} \mathrm{d}\sigma, \tag{2.30}$$

such that the evolution Equation (2.24) becomes

$$\frac{\partial q}{\partial \sigma^*} + q\frac{\partial q}{\partial \phi} = \left(\sum_{k=0}^{\infty} \widehat{a}_k(\sigma^*)\left(\frac{\partial q}{\partial \phi}\right)^k\right)\frac{\partial^2 q}{\partial \phi^2}, \tag{2.31}$$

where

$$\widehat{a}_k(\sigma^*) = \frac{a_k}{g(\sigma)} \exp\left\{(1-k)\int f(\sigma)\,\mathrm{d}\sigma\right\}.$$

In general, Equation (2.31) differs from the classical Burgers equation because of the coefficient at $\partial^2 q/\partial\phi^2$. In the particular cases (i) and (ii), this evolution equation becomes a generalized Burgers equation different from the well-known ones considered in [15, 16], because the coefficient at $\partial^2 q/\partial\phi^2$ also depends on $\partial q/\partial\phi$. In the particular case $\overline{\alpha} = \overline{\alpha}(\rho, e)$ and $\overline{\beta} = \overline{\beta}_0(\rho, e) + \overline{\beta}_1(\rho, e)$ I (i.e. Newtonian fluid) Equation (2.31) reduces to a generalized Burgers equation [15].

3. Sedov solution

Now we consider the particular class of non-Newtonian fluid [17, 18] for which the state equation for the pressure reads

$$p = (\Gamma - 1)\rho e, \tag{3.1}$$

which follows from (2.7) when $P = (\Gamma - 1)\zeta$ with Γ being the index of the fluid. In this case the quasi-linear hyperbolic system associated with (1.1) admits an invariant solution of the form (2.8) with

$$R_0 = \overline{\rho}_0\frac{\Gamma+1}{\Gamma-1}\xi, \qquad V_0 = \frac{2}{3\Gamma-1}\xi, \qquad E_0 = \frac{2}{(3\Gamma-1)^2}\xi^2, \qquad \gamma = \frac{3\Gamma-1}{\Gamma+1}. \tag{3.2}$$

This is the well-known Sedov solution which describes the flow behind a strong spherical shock wave generated by a point explosion in a medium having an initial mass distribution of the form

$$\rho_0(r) = \overline{\rho}_0 r^{(\Gamma-7)/(\Gamma+1)}. \tag{3.3}$$

In this case, we obtain from (2.22), evaluated for $\lambda = \lambda_2$, the following expressions for the characteristic rays

$$\xi = \xi_0 \exp(\overline{N}\sigma), \qquad \sigma = \tau, \qquad \overline{N} = \frac{1 - \Gamma + \sqrt{2\Gamma(\Gamma-1)}}{3\Gamma - 1} \tag{3.4}$$

with ξ_0 constant along the characteristic rays. Consequently, the transformations of variables (2.30) reduces to

$$q = \pi \exp(\overline{M}\sigma), \qquad \sigma^* = \frac{1}{\theta}\exp(\mu_1 - \mu_0 - \overline{M})\sigma, \tag{3.5}$$

where

$$\begin{aligned} \overline{M} &= \mu_1 + \frac{1 + \Gamma + 6\sqrt{2\Gamma(\Gamma-1)}}{2(3\Gamma-1)}, \qquad \theta = \frac{3\Gamma - 5 - 6\sqrt{2\Gamma(\Gamma-1)}}{\xi_0(\Gamma+1)\sqrt{2\Gamma(\Gamma-1)}} \\ \mu_1 - \mu_0 - \overline{M} &= \frac{3\Gamma - 5 - 6\sqrt{2\Gamma(\Gamma-1)}}{2(3\Gamma-1)}. \end{aligned} \tag{3.6}$$

Therefore, the evolution equation (2.31) may be rewritten as follows

$$\frac{\partial q}{\partial \sigma^*} + q\frac{\partial q}{\partial \phi} = S(\sigma^*)^Z \left(\sum_{k=0}^{\infty} \widetilde{a}_k(\sigma^*)\left(\frac{\partial q}{\partial \phi}\right)^k\right)\frac{\partial^2 q}{\partial \phi^2}. \tag{3.7}$$

with

$$\begin{aligned} S &= \frac{(\Gamma-1)(3\Gamma-1)}{\overline{\rho}_0(\Gamma+1)^2\xi_0^2\sqrt{2\Gamma(\Gamma-1)}}\theta^Z, \qquad Z = \frac{3\Gamma+7}{3\Gamma - 5 - 6\sqrt{2\Gamma(\Gamma-1)}}, \\ \widetilde{a}_k &= \frac{\xi_0^k}{k!}\frac{\sqrt{[2\Gamma(\Gamma-1)]^k}}{(3\Gamma-1)^k}(\theta\sigma^*)^{k-(k/(\mu_1-\mu_0-\bar{M}))}\left(\frac{\partial^{(k+1)}}{\partial u_r^{(k+1)}}(u_r\overline{\alpha} + \overline{\beta})\right)_0. \end{aligned} \tag{3.8}$$

4. Polynomial approximations

The constitutive Equations (1.3) are too complicated for us to proceed with the study of the evolution Equation (3.7). Hence we consider the polynomial approximation of various degrees in γ, as is customary in the treatment of problems in Stokesian fluid flow [3, pp. 168–171]. In this context we may assume the general nth-order theory in which the phenomenological coefficients $\overline{\beta}$ and $\overline{\alpha}$ are taken as polynomials of degree n, $n-1$ and $n-2$ in the invariants I, II, III, respectively.

Since the degrees of I, II and III in γ are 1, 2, 3, respectively, the most general polymomial approximation of degree n in γ is represented by the following constitutive relations

$$\begin{aligned} \overline{\beta} \;=\; & \sum_{n=1}^{k} b_n \mathrm{I}^n + \sum_{n=1}^{[k/2]} c_n \mathrm{II}^n + \sum_{n=1}^{[k/3]} e_n \mathrm{III}^n + \sum_{m=1}^{[(k-1)/2]}\left(\sum_{n=1}^{k-2m} f_n \mathrm{I}^n\right)\mathrm{II}^m \\ & + \sum_{m=1}^{[(k-1)/3]}\left(\sum_{n=1}^{k-3m} r_n \mathrm{I}^n\right)\mathrm{III}^m + \sum_{m=1}^{[(k-2)/3]}\left(\sum_{n=1}^{[(k-3m)/2]} g_n \mathrm{II}^n\right)\mathrm{III}^m, \end{aligned} \tag{4.1}$$

$$\begin{aligned} \overline{\alpha} \;=\; & \sum_{n=0}^{k-1} \widehat{b}_n \mathrm{I}^n + \sum_{n=1}^{[(k-1)/2]} \widehat{c}_n \mathrm{II}^n + \sum_{n=1}^{[(k-1)/3]} \widehat{e}_n \mathrm{III}^n + \sum_{m=1}^{[(k-2)/2]}\left(\sum_{n=1}^{k-1-2m} \widehat{f}_n \mathrm{I}^n\right)\mathrm{II}^m \\ & + \sum_{m=1}^{[(k-2)/3]}\left(\sum_{n=1}^{k-1-3m} \widehat{r}_n \mathrm{I}^n\right)\mathrm{III}^m + \sum_{m=1}^{[(k-3)/3]}\left(\sum_{n=1}^{[(k-1-3m)/2]} \widehat{g}_n \mathrm{II}^n\right)\mathrm{III}^m, \end{aligned} \tag{4.2}$$

where $[x]$ is the largest integer less than or equal to x. The coefficients occurring in (4.1) and (4.2) depend on the thermodynamical variables ρ and e. Furthermore the constitutive laws (4.1) and (4.2) guarantee that the condition (2.26) is fulfilled for $m = n$, so that the series occurring in the evolution equation (2.24) truncates after $n - 1$ terms.

In [3] the restrictions imposed by the Clausius–Duhem inequality on the polynomial approximation up to second order were studied. Here we limit our analysis to the case $n = 3$ where the constitutive laws (4.1) and (4.2) reduce, respectively, to

$$\overline{\beta} = b_1\mathrm{I} + b_2\mathrm{I}^2 + b_3\mathrm{I}^3 + c_1\mathrm{II} + e_1\mathrm{III} + f_1\mathrm{I}\,\mathrm{II}, \qquad \overline{\alpha} = \widehat{b}_0 + \widehat{b}_1\mathrm{I} + \widehat{b}_2\mathrm{I}^2 + \widehat{c}_1\mathrm{II}. \tag{4.3}$$

Owing to (4.3), the Clausius–Duhem inequality (2.29) reduces to

$$\widehat{b}_0\mathrm{II} + (\widehat{b}_1 + c_1)\mathrm{I}\,\mathrm{II} + (\widehat{b}_2 + f_1)\mathrm{I}^2\mathrm{II} + \widehat{c}_1\mathrm{II}^2 + b_1\mathrm{I}^2 + b_2\mathrm{I}^3 + b_3\mathrm{I}^4 + e_1\mathrm{I}\,\mathrm{III} \geqslant 0. \tag{4.4}$$

Contrary to the cases $n = 1, 2$, it appears that condition (4.4) cannot easily be handled and it can be satisfied by several choices of the coefficients involved therein. In particular (4.4) is satisfied if the following conditions hold

$$\begin{aligned} &\widehat{b}_0 \geqslant 0, \qquad \widehat{b}_2 \leqslant 0, \qquad \widehat{c}_1 = -\widehat{b}_2 \qquad b_1 \geqslant 0, \qquad b_2 = 0, \\ &b_3 = \frac{\widehat{b}_2}{3}, \qquad c_1 = -\widehat{b}_1, \qquad e_1 = \frac{8}{3}\widehat{b}_2, \qquad f_1 = -3\widehat{b}_2, \end{aligned} \tag{4.5}$$

so that Equation (3.7) assumes the form

$$\frac{\partial q}{\partial \sigma^*} + q\frac{\partial q}{\partial \phi} = S(\sigma^*)^Z \left(\widehat{b}_0 + b_1 + \frac{4}{3\Gamma - 1}(\theta\sigma^*)^{1/(\bar{M}-\mu_1+\mu_0)}\widehat{b}_1\right)\frac{\partial^2 q}{\partial \phi^2}. \tag{4.6}$$

Despite the nonlinear dependence of the viscous stress tensor upon $\boldsymbol{U}_r$, the evolution equation (4.6) obtained herein is a generalized Burgers equation with variable coefficient, which can be studied by means of the qualitative analysis developed by Scott [15, 16].

In passing we note that when $\widehat{b}_2 = 0$, owing to the restrictions (4.5), we recover exactly the second-order theory studied in [3], so that Equation (4.6) is also valid in the latter theory. In general for arbitrary coefficients satisfying the Clausius–Duhem inequality (2.29), the evolution equation (3.7) assumes the following form

$$\frac{\partial q}{\partial \sigma^*} + q\frac{\partial q}{\partial \phi} = S(\sigma^*)^Z \left(C_0(\sigma^*) + C_1(\sigma^*)\frac{\partial q}{\partial \phi} + C_2(\sigma^*)\left(\frac{\partial q}{\partial \phi}\right)^2\right)\frac{\partial^2 q}{\partial \phi^2}, \tag{4.7}$$

where

$$\begin{aligned} C_0(\sigma^*) &= h_0 + h_1\sigma^{*(1/(\bar{M}+\mu_0-\mu_1))} + h_2\sigma^{*(2/(\bar{M}+\mu_0-\mu_1))}, \\ C_1(\sigma^*) &= (\widetilde{h}_0 + \widetilde{h}_1\sigma^{*(1/(\bar{M}+\mu_0-\mu_1))})\sigma^{*(1+(1/(\bar{M}+\mu_0-\mu_1)))}, \\ C_2(\sigma^*) &= \widehat{h}_0\sigma^{*(2+(2/(\bar{M}+\mu_0-\mu_1)))} \end{aligned} \tag{4.8}$$

with

$$
\begin{aligned}
&h_0 = \widehat{b}_0 + b_1, \qquad h_1 = \frac{4}{3\Gamma - 1}(3b_2 + 2\widehat{b}_1 + c_1)\theta^{1/(\bar{M}+\mu_0-\mu_1)},\\
&h_2 = \frac{4}{(3\Gamma-1)^2}(5\widehat{c}_1 + 3e_1 + 27b_3 + 9f_1 + 15\widehat{b}_2)\theta^{2/(\bar{M}+\mu_0-\mu_1)},\\
&\widetilde{h}_0 = \frac{2\xi_0\sqrt{2\Gamma(\Gamma-1)}}{3\Gamma-1}(b_2 + \widehat{b}_1 + c_1)\theta^{1+(1/(\bar{M}+\mu_0-\mu_1))},\\
&\widetilde{h}_1 = \frac{4\xi_0\sqrt{2\Gamma(\Gamma-1)}}{(3\Gamma-1)^2}(3\widehat{c}_1 + 3e_1 + 9b_3 + 5f_1 + 7\widehat{b}_2)\theta^{1+(2/(\bar{M}+\mu_0-\mu_1))},\\
&\widehat{h}_0 = \frac{6\xi_0^2\Gamma(\Gamma-1)}{(3\Gamma-1)^2}(\widehat{c}_1 + e_1 + b_3 + f_1 + \widehat{b}_2)\theta^{2+(2/(\bar{M}+\mu_0-\mu_1))}.
\end{aligned}
\tag{4.9}
$$

We remark that with respect to other classes of modified Burgers equations already known in the literature, the one obtained herein exhibits a coefficients of $\partial^2 q/\partial\phi^2$ which depends upon $\partial q/\partial\phi$.

5. Similarity analysis

Unfortunately the obtained generalized Burgers equation (4.7) does not lend itself to exact analytic treatment. In fact, in [19] it was shown that only generalized Burgers equation for which a Backlund transformation exists is the classical one with an additional source term. Therefore, owing to the lack of linearizing Backlund transformations, one must deal directly with generalized Burgers equations by taking recourse to matched asymptotic analysis, similarity analysis or numerical methods.

Within the latter theoretical framework we require the invariance of the Equation (4.7) with respect to an infinitesimal group of transformations [7–9]

$$
\begin{aligned}
&\sigma^{*\prime} = \sigma^* + \omega X(\sigma^*, \phi, q), \qquad \phi' = \phi + \omega F(\sigma^*, \phi, q),\\
&q' = q + \omega Q(\sigma^*, \phi, q).
\end{aligned}
\tag{5.1}
$$

Along the lines of a well-established procedure [7–9], several possibilities arise for the generators of the group

$$
\text{(i)} \quad X = 0, \qquad F = q_0\sigma^* + \overline{q}_0, \qquad Q = q_0, \tag{5.2}
$$

where q_0 and $\overline{q}_0$ are arbitrary constants. Consequently, in this case we obtain a similarity solution of the type

$$
q = \frac{q_0\phi + \widehat{q}_0}{q_0\sigma^* + \overline{q}_0}, \quad \widehat{q}_0 = \text{const.} \tag{5.3}
$$

(ii)

$$
\begin{aligned}
&X = q_2(\sigma^*)^2 + 2q_3\sigma^*, \qquad F = q_0\sigma^* + (q_2\sigma^* + 3q_3)\phi + \overline{q}_0,\\
&Q = (q_3 - q_2\sigma^*)q + q_2\phi + q_0
\end{aligned}
\tag{5.4}
$$

with

$$\frac{11-9\Gamma}{3\Gamma-5-6\sqrt{2\Gamma(\Gamma-1)}}=2, \qquad h_0=h_1=\widetilde{h}_0=0, \qquad \widetilde{h}_1=-2h_2=-2\widehat{h}_0. \tag{5.5}$$

Since $(5.5)_1$ is not satisfied for any value of the fluid index Γ, the present case is not considered in the following

$$\text{(iii)} \quad X=(f_2-q_3)\sigma^*, \qquad F=q_0\sigma^*+f_2\phi+\overline{q}_0, \qquad Q=q_3q+q_0, \tag{5.6}$$

where q_0, $\overline{q}_0$, f_2 and q_3 are arbitrary constants with

$$(8-6\Gamma+3\sqrt{2\Gamma(\Gamma-1)})f_2-3(1-\Gamma-\sqrt{2\Gamma(\Gamma-1)})q_3=0 \qquad h_0=h_1=\widetilde{h}_0=0. \tag{5.7}$$

Comparison of (4.9) and $(5.7)_2$ with use of the Clausius–Duhem inequality, give rise to the further restrictions

$$\widehat{b}_0=b_1=b_2=c_1=\widehat{b}_1=0, \qquad b_3=-\frac{\widehat{b}_2+f_1}{6}, \qquad e_1=-\frac{4}{3}(\widehat{b}_2+f_1), \qquad \widehat{c}_1\geqslant 0, \qquad |\widehat{b}_2+f_1|\leqslant 2\widehat{c}_1, \tag{5.8}$$

whereupon the constitutive relations (4.3) become

$$\overline{\beta}=b_3\mathrm{I}^3+e_1\mathrm{III}+f_1\mathrm{I\,II}, \qquad \overline{\alpha}=\widehat{b}_2\mathrm{I}^2+\widehat{c}_1\mathrm{II}. \tag{5.9}$$

Next we consider the following two cases

$(\text{iii})_1$ if $q_3\neq 0$, the solution of (4.7) is given by

$$q=-\frac{q_0}{q_3}+(\sigma^*)^{q_3/(f_2-q_3)}\widehat{Q}(\chi), \qquad \chi=(\sigma^*)^{-(f_2/(f_2-q_3))}\left(\phi+\frac{q_0}{q_3}\sigma^*+\frac{\overline{q}_0}{f_2}\right) \tag{5.10}$$

with $\widehat{Q}$ a solution of the second-order differential equation

$$S(h_2+\widetilde{h}_1\widehat{Q}'+\widehat{h}_0\widehat{Q}'^2)\widehat{Q}''=\left(\widehat{Q}-\frac{f_2}{f_2-q_3}\chi\right)\widehat{Q}'+\frac{q_3}{f_2-q_3}\widehat{Q}. \tag{5.11}$$

$(\text{iii})_2$ if $q_3=0$, the solution of (4.7) is given by

$$q=\frac{q_0}{f_2}\log\sigma^*+\widetilde{Q}(\widetilde{\chi}), \qquad \widetilde{\chi}=\frac{\phi}{\sigma^*}-\frac{q_0}{f_2}\log\sigma^*+\frac{\overline{q}_0}{f_2\sigma^*}, \tag{5.12}$$

where $\widetilde{Q}$ is a solution of the following second order differential equation

$$S(h_2+\widetilde{h}_1\widetilde{Q}'+\widehat{h}_0\widetilde{Q}'^2)\widetilde{Q}''=\left(\widetilde{Q}-\widetilde{\chi}-\frac{q_0}{f_2}\right)\widetilde{Q}'+\frac{q_0}{f_2}\widetilde{Q}. \tag{5.13}$$

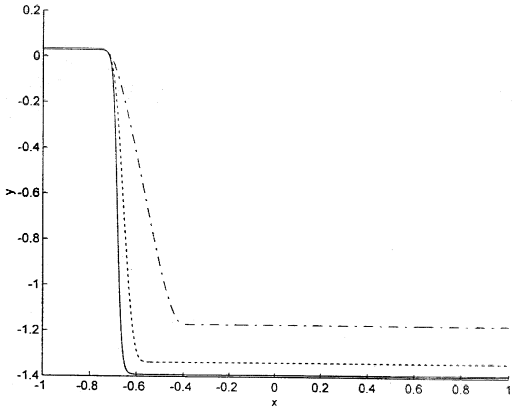

Figure 1. Comparison between self-similar solutions of the classical Burgers equation for cylindrical waves (solid line) and of the modified ones corresponding to the initial conditions $y(-1) = 0{\cdot}03$, $y'(-1) = -10^{-9}$ and the choice $Sh_2/b^2 = 0{\cdot}007$. The dashed profile (----) is obtained for $S\widetilde{h}_1/b^2 = -0{\cdot}001$, $\widehat{h}_0 = 0$; the dashdot one (–––) i s obtained for $S\widetilde{h}_1/b^2 = -0{\cdot}001$, $S\widehat{h}_0/b^2 = 0 \cdot 004$.

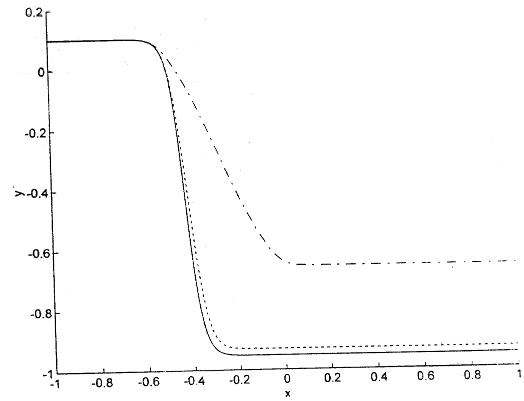

Figure 2. Comparison between self-similar solutions of the classical Burgers equation for cylindrical waves (solid line) and of the modified ones corresponding to the initial conditions $y(-1) = 0{\cdot}1$, $y'(-1) = -10^{-7}$ and the choice $Sh_2/b^2 = 0{\cdot}02$. The dashed profile (----) is obtained for $S\widetilde{h}_1/b^2 = -0{\cdot}001$, $\widehat{h}_0 = 0$; the dashdot one (–––) is obtained for $S\widetilde{h}_1/b^2 = -0{\cdot}001$, $S\widehat{h}_0/b^2 = 0{\cdot}05$.

Taking into account the conditions $(5.7)_2$, we find that the generalized Burgers equation (4.7) in the present case reduces to

$$\frac{\partial q}{\partial \sigma^*} + q\frac{\partial q}{\partial \phi} = S(\sigma^*)^{(11-9\Gamma)/(3\Gamma-5-6\sqrt{2\Gamma(\Gamma-1)})}\left(h_2 + \widetilde{h}_1\sigma^*\frac{\partial q}{\partial \phi} + \widehat{h}_0(\sigma^*)^2\left(\frac{\partial q}{\partial \phi}\right)^2\right)\frac{\partial^2 q}{\partial \phi^2} \tag{5.14}$$

When $q_3 = 0$, it follows from $(5.7)_1$

$$8 - 6\Gamma + 3\sqrt{2\Gamma(\Gamma-1)} = 0,$$

whereupon the resulting Equation (5.14) generalizes the Burgers equation for cylindrical waves owing to the powers of $\partial q/\partial\phi$ occurring in the coefficient at $\partial^2 q/\partial\phi^2$.

As is well-known, for cylindrical waves there exists an exact solution which is of shock type [20, pp. 70–73]. This exact solution is defined implicitly in terms of a similarity variable through two integral expressions which have no representation in terms of known functions; such a solution belongs to the class (5.12) characterized by our analysis when $q_0 = \overline{q}_0 = 0$.

Next, by assuming the parameters $\widetilde{h}_1$, $\widehat{h}_0$ in Equation (5.14) (or (5.13)) to be small, we would expect that the qualitative behaviour of the solution of (5.14) and of the solution of the corresponding Burgers' equation for cylindrical waves do not differ too much one from another. We will show that below by integrating numerically the ODE (5.13). The different wave profiles which are considered correspond to several values of the parameters involved in (5.14).

The solution is shown in Figures 1 and 2, where the scaled variable $y = \widetilde{Q}/b$ is plotted against $x = \widetilde{\chi}/b$ with $b \gg 1$.

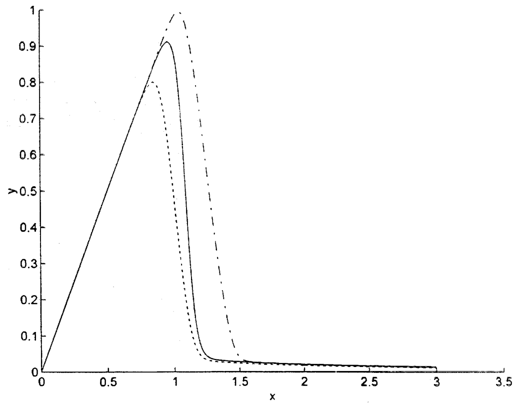

Figure 3. Comparison between the triangular wave solutions of the classical Burgers equation for plane waves (solid line) and of the modified ones corresponding to the initial conditions $y(0) = 2 \cdot 10^{-7}$, $y'(0) = 0{\cdot}999995$ and the choice $Sh_2/b^2 = 0{\cdot}02$. The dashed profile (----) is obtained for $S\widetilde{h}_1/b^2 = -0{\cdot}0054$, $\widehat{h}_0 = 0$; the dashdot one (– - – -) is obtained for $S\widetilde{h}_1/b^2 = -0{\cdot}0054$, $S\widehat{h}_0/b^2 = 0{\cdot}01$.

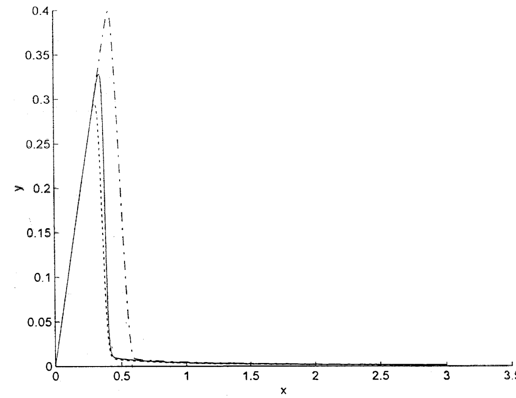

Figure 4. Comparison between the triangular wave solutions of the classical Burgers equation for plane waves (solid line) and of the modified ones corresponding to the initial conditions $y(0) = 2 \cdot 10^{-7}$, $y'(0) = 0{\cdot}999996$ and the choice $Sh_2/b^2 = 0{\cdot}002$. The dashed profile (----) is obtained for $S\widetilde{h}_1/b^2 = -0{\cdot}0006$, $\widehat{h}_0 = 0$; the dashdot one (– - – -) is obtained for $S\widetilde{h}_1/b^2 = -0{\cdot}0006$, $S\widehat{h}_0/b^2 = 0{\cdot}002$.

Finally, let us consider a fluid with the index $\Gamma = 11/9$. Here the resulting Equation (5.14), because of the parameters $\widetilde{h}_1$, $\widehat{h}_0$ involved in the coefficient at $\partial^2 q/\partial\phi^2$, generalizes the classical Burgers equation for plane waves. As is well-known, the latter equation ($\widetilde{h}_1 = \widehat{h}_0 = 0$) admits the very special single hump solution [21, pp. 101–107] which is of similarity type; it is recovered from the class (5.10) when $q_0 = \overline{q}_0 = 0$. Such a solution for $Sh_2 \ll 1$ (e.g. large Reynolds numbers R) exhibits a triangular wave profile.

As for the cylindrical waves we considered above, also in the present case, by assuming $\widetilde{h}_1$ and $\widehat{h}_0$ to be small, a numerical integration of the ODE (5.11) ruling the similarity solutions of (5.14) permits to characterize a profile of triangular form.

In Figures 3 and 4 the scaled variable $y = \widehat{Q}/\sqrt{\mathsf{R}}$ is plotted against $x = \chi/\sqrt{\mathsf{R}}$.

6. Conclusions and remarks

In this paper we have considered the dissipative system describing the spherical symmetric motion of a non-Newtonian compressible fluid in order to investigate the point-explosion problem. We note that the model under consideration may serve to describe, from a macroscopic point of view, a gas in which long-chain molecules are dissolved. Such a situation occurs in the macroscopic behaviour of many real applications like farmaceutical aerosol, environmental aerosol (sandstorm, soot) and industrial aerosol (smog, fuel).

We have studied the effects of nonlinearity and dissipation involved in the model by assuming an asymptotic expansion around a similarity solution of the associated hyperbolic system. In the far field approximation we deduced an evolution equation which is different from the usual Burgers-like equation one obtains for dissipative models owing to the occurrence of the series-like coefficient of the second-order derivative. A consistency argument for the obtained equation led us to require some conditions to be satisfied by the viscosity coefficients. These restrictions allowed us to select classes of functional forms for the phenomenological coefficients therein involved. We note that, as far as we know, in the case of non-Newtonian compressible fluids, very little information about these coefficients exists. Different cases, where the series-like coefficient occurring in the evolution equation truncates, were considered. Among other things we recovered the Stokesian fluid of nth-order as a particular case by assuming the polynomial approximation to hold, so that we could deduce a generalized Burgers equation with variable coefficient which is a polynomial in the first-order derivatives.

In the particular case of a Stokesian fluid of third-order, making use of the similarity analysis, we obtained several classes of solutions to the evolution equation in point. Finally, via a numerical integration, we made a comparison between the wave profiles obtained herein and those related to the classical Burgers equation for cylindrical and for plane waves, respectively.

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